# Texture Characterisation Using Constrained Optimisation Techniques with Application to Mammography

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**Abstract**

*Texture characterisation problem is attempted using two-dimensional ($2-D$) autoregressive (AR) modelling under the assumption that different textures can be described by different sets of $2-D$ AR model coefficients. The AR model coefficients are estimated using a proposed technique which relates the conventional Yule-Walker system of equations with the Yule-Walker system of equations in the third-order statistical domain through a constrained optimisation formulation with equality constraints. The method is applied to characterise textures in mammograms.*

## 1 Introduction

Texture characterisation has been widely used in different kinds of images including medical images and SAR (synthetic aperture radar) images [1][2]. Many different methods have been used for texture characterisation purposes. In this paper, the problem is attempted by treating the image as the output of two-dimensional ($2-D$) autoregressive (AR) model, therefore in order to characterise textures, we simply estimate the AR model coefficients of the given images or the area of interest.

Two of the most popular methods in the literature for estimating AR model coefficients are the Yule-Walker system of equations (YW) and the Yule-Walker system of equations in the third-order statistical domain (YWT). The first method is able to estimate AR model coefficients only when the SNR (signal-to-noise ratio) is high, i.e. the external Gaussian noise is small compared to the signal and the variance of estimated AR model coefficients arisen from a number of realisations is lower compared to the later method. The YWT method employs the third-order moments of the samples, therefore the external Gaussian noise can be eliminated in estimations, however, the variance of estimated values is higher [6][7]. Our proposed method relates the YWT method with the YW method through a constrained optimisation formulation with equality constraints. The conventional methods may be found in Section 2 and the proposed method is in Section 3 followed by the simulation results of both synthetic images and mammograms in Section 4.

## 2 Preliminaries

### 2.1 Two-Dimensional Autoregressive Model

Let us consider a digitised image $x$ of size $M \times N$. Each pixel of $x$ is characterised by its location $[m, n]$ and can be represented as $x[m, n]$, where $1 \leq m \leq M$, $1 \leq n \leq N$ and $x[m, n]$ is a positive intensity (gray level) associated with it. A two-dimensional $(2-D)$ autoregressive (AR) model is defined as [4]

$$x[m,n] = -\sum_{i=0}^{p_1}\sum_{j=0}^{p_2} a[i,j]x[m-i,n-j] + w[m,n], \quad (1)$$

where $[i, j] \neq [0, 0]$, $a[i, j]$ is the AR model coefficient, $w[m, n]$ is the input driving noise, $x[m, n]$ is the output and $p_1 \times p_2$ is the order of the AR model.

The driving noise, $w[m, n]$, is non-Gaussian and assumed to be zero-mean, i.e., $E\{w[m, n]\} = 0$, where $E\{\cdot\}$ is the expectation operation. The AR model coefficient $a[0, 0]$ is assumed to be 1 for scaling purposes, therefore we have $[(p_1 + 1)(p_2 + 1) - 1]$ unknown coefficients to solve.

An additional zero-mean Gaussian noise, $v[m, n]$, with variance equal to unity, is added onto the system. Mathematically the new system can be written as

$$y[m, n] = x[m, n] + v[m, n]. \quad (2)$$

The signal-to-noise ratio (SNR) of the system is calculated by

$$SNR = 10log_{10}\frac{\sigma_x^2}{\sigma_v^2} \quad dB \quad (3)$$

where $\sigma_x^2$ is the variance of the signal and $\sigma_v^2$ is the variance of the noise.

### 2.2 Yule-Walker System of Equations

The YW method for the quarter-plane (QP) model, i.e., the region of support of AR model parameters is in the quarter plane only, can be written in matrix form as [4]

$$\underline{\boldsymbol{R}}_{yy}\underline{\boldsymbol{a}}_1 = \underline{\boldsymbol{h}}, \tag{4}$$

where $\underline{\boldsymbol{R}}_{yy}$ is a $(p_1+1)(p_2+1) \times (p_1+1)(p_2+1)$ matrix of second-order moment samples and $\underline{\boldsymbol{a}}_1$ and $\underline{\boldsymbol{h}}$ are both $(p_1+1)(p_2+1) \times 1$ vectors.

More explicitly, (4) can be rewritten as in (5) on the next page. In (5), $\boldsymbol{a}[i] = [a[i,0], a[i,1], \cdots, a[i,p_2]]^T$ is of dimension $(p_2+1) \times 1$, $T$ denotes transpose,
$\boldsymbol{h}_1 = [1, 0, \cdots, 0]^T$ is of dimension $(p_2+1) \times 1$,
$\boldsymbol{0} = [0, 0, \cdots, 0]$ is of dimension $(p_2+1) \times 1$, and
$\boldsymbol{R}_{yy}[i] =$

$$\begin{pmatrix} r_{yy}[i,0] & r_{yy}[i,-1] & \cdots & r_{yy}[i,-p_2] \\ r_{yy}[i,1] & r_{yy}[i,0] & \cdots & r_{yy}[i,-(p_2-1)] \\ \vdots & \vdots & \ddots & \vdots \\ r_{yy}[i,p_2] & r_{xx}[i,p_2-1] & \cdots & r_{yy}[i,0] \end{pmatrix}$$

is of dimension $(p_2+1) \times (p_2+1)$.

The second-order moment sample $r_{yy}[i,j]$ is defined as

$$r_{yy}[i,j] = E\{y[m,n]y[m+i,n+j]\}. \tag{6}$$

Since the input variance $\sigma_w^2$ is unknown, the first equation in (5) may be eliminated. The coefficient $a[0,0]$ is assumed to be 1, so we can move the first column of the matrix of the remaining system to the right-hand side of the system. Let us write the revised system as

$$\underline{\boldsymbol{R}}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{r}}, \tag{7}$$

where $\underline{\boldsymbol{R}}$ is a $(p_1p_2+p_1+p_2) \times (p_1p_2+p_1+p_2)$ matrix of second-order moment samples and $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{r}}$ are both $(p_1p_2+p_1+p_2) \times 1$ vectors.

### 2.3 Yule-Walker System of Equations in the Third-Order Statistical Domain

The YW method introduced in the previous section is able to estimate the AR model coefficients only when the system SNR is large [6]. When the SNR is small, the estimation results are influenced by the large Gaussian noise. In the literature, the YWT method has been used to solve this problem. The equations may be written in matrix form as [6][7]

$$\underline{\boldsymbol{C}}_{yy}\underline{\boldsymbol{a}}_1 = -\underline{\boldsymbol{c}}_{yy}. \tag{8}$$

More explicitly, (8) can be rewritten as in (9) on the next page. In (9) $\boldsymbol{a}[i] = [a[i,0], a[i,1], \cdots, a[i,p_2]]^T$ is of dimension $(p_2+1) \times 1$, $\boldsymbol{h}_1 = [1, 0, \cdots, 0]^T$ is of dimension $(p_2+1) \times 1$, $\boldsymbol{0} = [0, 0, \cdots, 0]^T$ is of dimension $(p_2+1) \times 1$ and the matrix $\underline{\boldsymbol{C}}_{3y}[i]$ of dimensions $(p_2+1) \times (p_2+1)$ is shown on the next page.

The third-order moment sample for a zero-mean process is estimated by [6]
$C_{3y}([i_1, j_1], [i_2, j_2]) =$

$$E\{y[m,n]y[m+i_1,n+j_1]y[m+i_2,n+j_2]\} \tag{10}$$

The skewness of the driving input is also unknown, therefore we can further simplify the equations applying the same rule mentioned at the end of last section. The equations can now be written as

$$\underline{\boldsymbol{C}}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{c}} \tag{11}$$

for model order $p_1 \times p_2$, where $\underline{\boldsymbol{C}}$ is a $(p_1p_2+p_1+p_2) \times (p_1p_2+p_1+p_2)$ matrix of third-order moment samples and $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{c}}$ are both $(p_1p_2+p_1+p_2) \times 1$ vectors.

## 3 Proposed Method

The proposed method relates (11) to (7) through a constrained optimisation formulation, which can be written mathematically as

$$minimise \sum_{i=1}^{W-1} (R_i\underline{\boldsymbol{a}} + r_i)^2,$$

subject to

$$\underline{\boldsymbol{C}}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{c}} \tag{12}$$

where $W$ is the number of rows in matrix $\underline{\boldsymbol{R}}$ in (7),
$R_i$ is the $i$th row of the matrix $\underline{\boldsymbol{R}}$ in (7),
$\underline{\boldsymbol{a}}$ is the vector of unknown AR model parameters,
$r_i$ is the $i$-th element of the vector $\underline{\boldsymbol{r}}$ in (7), and
$\underline{\boldsymbol{C}}$ and $\underline{\boldsymbol{c}}$ are derived in (11).

(12) is solved using sequential quadratic programming (SQP) [3].

## 4 Simulation Results

### 4.1 Synthetic Images

Synthetic images generated from the following $1 \times 1$ stable and separable AR model are used for simulation purposes. The $2-D$ stable AR model coefficients are obtained from $\underline{a} = \underline{a}_1^T \times \underline{b}_1$, where $\underline{a}_1$ and $\underline{b}_1$ are both stable $1-D$ AR model coefficients.
$x[m,n] = -0.25x[m-1,n-1] - 0.5x[m-1,n] - 0.5x[m,n-1] + w[m,n]$
The driving noise, $w[m,n]$, is zero-mean exponentially-distributed. Additional Gaussian

$$
\begin{pmatrix}
\boldsymbol{R}_{yy}[0] & \boldsymbol{R}_{yy}[-1] & \cdots & \boldsymbol{R}_{yy}[-p_1] \\
\boldsymbol{R}_{yy}[1] & \boldsymbol{R}_{yy}[0] & \cdots & \boldsymbol{R}_{yy}[-(p_1-1)] \\
\vdots & \vdots & \ddots & \vdots \\
\boldsymbol{R}_{yy}[p_1] & \boldsymbol{R}_{yy}[p_1-1] & \cdots & \boldsymbol{R}_{yy}[0]
\end{pmatrix}
\begin{pmatrix} \boldsymbol{a}[0] \\ \boldsymbol{a}[1] \\ \vdots \\ \boldsymbol{a}[p_1] \end{pmatrix}
=
\begin{pmatrix} \sigma_w^2 \boldsymbol{h}_1 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}
\tag{5}
$$

$$
\begin{pmatrix}
\underline{\boldsymbol{C}}_{3y}[0] & \underline{\boldsymbol{C}}_{3y}[-1] & \cdots & \underline{\boldsymbol{C}}_{3y}[-p_1] \\
\underline{\boldsymbol{C}}_{3y}[1] & \underline{\boldsymbol{C}}_{3y}[0] & \cdots & \underline{\boldsymbol{C}}_{3y}[-(p_1-1)] \\
\vdots & \vdots & \ddots & \vdots \\
\underline{\boldsymbol{C}}_{3y}[p_1] & \underline{\boldsymbol{C}}_{3y}[p_1-1] & \cdots & \underline{\boldsymbol{C}}_{3y}[0]
\end{pmatrix}
\begin{pmatrix} \boldsymbol{a}[0] \\ \boldsymbol{a}[1] \\ \vdots \\ \boldsymbol{a}[p_1] \end{pmatrix}
= \gamma_w
\begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}
\tag{9}
$$

$$
\underline{\boldsymbol{C}}_{3y}[i] =
\begin{pmatrix}
C_{3y}([i,0],[i,0]) & C_{3y}([i,-1],[i,-1]) & \cdots & C_{3y}([i,-p_2],[i,-p_2]) \\
C_{3y}([i,1],[i,1]) & C_{3y}([i,0],[i,0]) & \cdots & C_{3y}([i,-(p_2-1)],[i,-(p_2-1)]) \\
\vdots & \vdots & \ddots & \vdots \\
C_{3y}([i,p_2],[i,p_2]) & C_{3y}([i,p_2-1],[i,p_2-1]) & \cdots & C_{3y}([i,0],[i,0])
\end{pmatrix}
$$

noise, $v[m,n]$, with zero-mean, is added onto $x[m,n]$ to yield

$$y[m,n] = x[m,n] + v[m,n].$$

The variance of $v[m,n]$ is adjusted so that SNR is equal to 5 dB for very noisy case and 30 dB for almost noise-free case. Simulation results obtained from the new method may be found in Table 1 for both SNR equal to 5 dB and 30 dB. One hundred realisations were taken, so that the variances could be calculated.

| Para-meter | Real Value | Estimated value | Variance ($10^{-3}$) | Estimated value | Variance ($10^{-3}$) |
|---|---|---|---|---|---|
| SNR | | 5 dB | | 30 dB | |
| $a[0,1]$ | 0.5 | 0.5024 | 0.1898 | 0.4998 | 0.06697 |
| $a[1,0]$ | 0.5 | 0.5021 | 0.3525 | 0.4997 | 0.04339 |
| $a[1,1]$ | 0.25 | 0.2553 | 0.7589 | 0.2498 | 0.1191 |
| Relative Error | | 0.03470 | | 0.01464 | |

Table 1: The results arisen from constrained optimisation with equality constraints for $2-D$ AR model coefficient estimation.

### 4.2 Mammography

Mammograms play an important role in breast cancer screening. We pay attention to the changes in AR model coefficients of the problematic area and its neighbourhood.

For each mammogram with mass, we form a neighbourhood around the square area of where the mass is and then estimate the AR model coefficients of each block in the neighbourhood. The order of the AR model is assumed to be $1 \times 1$.

An example of mammograms with a malignant mass taken here is the mdb023 from the database [5], which is shown in Figure 2 with the mass marked. The background tissue of this mammogram is fatty-glandular and the malignant mass is centred at the position [538, 681] with the radius 29 (pixels). The origin of the co-ordinate system is the bottom-left corner. We estimate the AR model coefficients of each block in neighbourhood as shown in Figure 1 and calculate the"degree of symmetry" by $a[1,1] - a[0,1] * a[1,0]$. The estimated AR model coefficients for Block 1, 2 and 3 can be found in Table 2, for Block 4, P and 5 in Table 3 and for Block 6, 7 and 8 in Table 4.

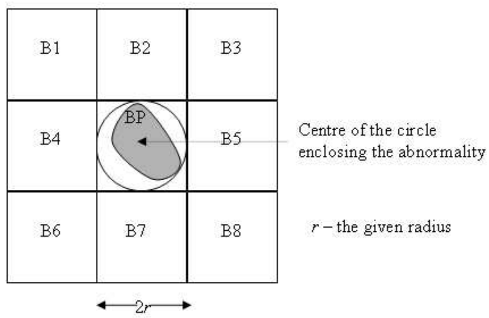


Figure 1: The mass and its neighbourhood.

## 5 Summary and Conclusions

A method is proposed using both the conventional Yule-Walker system of equations and the Yule-Walker


Figure 2: The mammogram with the mass marked: mdb023

| | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $a[0,1]$ | -0.9104 | 7.0822 | 2.4479 |
| $a[1,0]$ | -0.9643 | -0.3474 | -1.3119 |
| $a[1,1]$ | 0.8759 | -7.7154 | -2.3145 |
| Degree of symmetry | 0.0020 | 5.2553 | -1.0768 |

Table 2: The AR model coefficients for blocks 1-3 of pixels in mdb23.

| | $B_4$ | $B_P$ | $B_5$ |
|---|---|---|---|
| $a[0,1]$ | -1.3301 | -1.0403 | -1.1647 |
| $a[1,0]$ | -0.9151 | -1.0291 | -0.7296 |
| $a[1,1]$ | 1.2453 | 1.0696 | 0.8944 |
| Degree of symmetry | -0.0282 | 0.0010 | -0.0446 |

Table 3: The AR model coefficients for blocks 4, P and 5 of pixels in mdb23.

system of equations in the third-order statistical domain through a constrained optimisation formulation. The method is able to estimate $2-D$ AR model coefficients in both low and high SNR systems. Further simulations showed that the variances of the estimated coefficients arisen from the proposed method is lower than from the Yule-Walker system of equations in the third-order statistical domain. The method is also applied to characterise textures in the tumour

| | $B_6$ | $B_7$ | $B_8$ |
|---|---|---|---|
| $a[0,1]$ | -0.1890 | -0.8935 | 0.6717 |
| $a[1,0]$ | -0.8346 | -1.1864 | -1.0080 |
| $a[1,1]$ | 0.0255 | 1.0800 | -0.6707 |
| Degree of symmetry | 0.1322 | -0.0199 | -0.0016 |

Table 4: The AR model coefficients for blocks 6-8 of pixels in mdb23.

and its neighbourhood in mammograms. The problematic area can be presented by a set of symmetrical AR model coefficients, as the degree of symmetry is much lower in the problematic block compared to other blocks.

## References


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